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First-Order Transitions in One-Dimensional Systems with Local Couplings

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We point out the existence of first-order phase transitions in a family of one-dimensional classical spin systems. The relevant features of such models are that they involve only local (but complex) interactions and that the corresponding transfer matrices are self-adjoint operators. Moreover, for a wide range of coupling parameters the models satisfy the reflection positivity condition. The generalization for continuous spin systems enjoys similar properties.

KEY WORDS: Phase transitions; reflection positivity; clock models.

It is well known that phase transitions cannot occur in one-dimensional classical systems with real interactions which are *not too long-ranged.* More precisely, if the maximum interaction energy of the spins $\{\sigma_i\}_{-L\leqslant i\leqslant L}$ with the remaining spins $\{\sigma_i\}_{|i|>L}$ is bounded as $L\to\infty$, then there is a unique infinite-volume Gibbs measure at all temperatures (see ref. 1 for a recent review). In particular, for pair interactions $J(x-y)$ this happens whenever $\sum_x |x|\,|J(x)|<\infty$. Theorems of this kind can be proven by several different methods, including Perron–Frobenius arguments,[2] entropy–energy arguments,[3] and a very simple finite-energy argument.[4]

Therefore, in order to construct a one-dimensional classical spin system having a phase transition, it is necessary to violate at least one of the hypotheses of the uniqueness theorems, namely reality or short-rangedness. One possibility is to make the interaction *long-ranged,* such as $J(x)\sim |x|^{-\alpha}$ with $1<\alpha\leqslant 2$. In this case it is possible to establish rigorously the existence of a phase transition: see refs. 1 and 5–7 for the case $1<\alpha<2$, and ref. 8 for the more delicate and intriguing[9] borderline case $\alpha=2$. In

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some such cases the reflection-positivity property also holds: for example, it holds for $J(x) = A(|x| + b)^{-\alpha}$ for any A, $b \geqslant 0$ and $\alpha > 0$.[10]

Another possibility, less commonly considered, is to make the interaction energy complex. For example, if one introduces a pure imaginary external magnetic field, then there is a phase transition at low enough temperatures (Lee–Yang singularity).[11] However, in this example the transfer matrix is not self-adjoint, and in particular the interaction is not reflection-positive, which prevents the connection with quantum systems in the continuum limit. In this note we introduce a family of one-dimensional classical spin models with a *local, complex* interaction which have a self-adjoint (Hermitian) transfer matrix, satisfy reflection positivity, and have a series of first-order phase transitions. A similar complex interaction with self-adjoint transfer matrix has been previously considered by Lebowitz and Gallavotti[12] for the Ising model. However, such a model exhibits a conventional critical behavior and has no phase transition for finite temperature.

We consider a classical spin variable $\mathbf{s}_n$ fluctuating on the unit circle among the q-roots of unity,

$$\mathbf{s}_n = \left(\cos\frac{2\pi p_n}{q}, \sin\frac{2\pi p_n}{q}\right), \qquad p_n = 0, ..., q-1 \tag{1}$$

with the following Hamiltonian:

$$\begin{aligned} \mathscr{H} &= -\sum_{n=1}^{N} (J\mathbf{s}_n \cdot \mathbf{s}_{n+1} - i\varepsilon \mathbf{s}_n \times \mathbf{s}_{n+1}) \\ &= -\sum_{n=1}^{N} \left[J\cos\frac{2\pi}{q}(p_n - p_{n+1}) + i\varepsilon \sin\frac{2\pi}{q}(p_n - p_{n+1}) \right] \end{aligned} \tag{2}$$

where the coupling constants J and ε are real numbers. For $\varepsilon = 0$ the model reduces to the nearest neighbor $\mathbf{Z}_q$-states clock model ($q = 2$ corresponds to the Ising model) which has no phase transition in one dimension for nonvanishing temperature ($\beta < \infty$).

The Hamiltonian (2) is translation invariant and involves only local couplings. The second term in (2) is not invariant under inversion of space orientation. The partition function of the system is given by

$$Z(\beta, J, \varepsilon) = \sum_{\{s\}} \prod_{n=1}^{N} \exp(\beta J \mathbf{s}_n \cdot \mathbf{s}_{n+1} - i\beta\varepsilon \mathbf{s}_n \times \mathbf{s}_{n+1}) \tag{3}$$

We consider a finite-space volume with N sites and periodic boundary conditions, $\mathbf{s}_{N+1} = \mathbf{s}_1$. The transfer matrix T is a self-adjoint operator (i.e., a Hermitian $q \times q$ matrix) for any real values of the coupling constants J and ε.

Let us consider the particular cases $q = 3, 4$ for simplicity. The eigenvalues of the transfer matrix T for the 3-state model are

$$\begin{aligned} \lambda_0 &= e^{J\beta} + 2e^{-J\beta/2} \cos(\beta\varepsilon \sqrt{3}/2) \\ \lambda_1 &= e^{J\beta} - 2e^{-J\beta/2} \cos(\beta\varepsilon \sqrt{3}/2 + \pi/3) \\ \lambda_2 &= e^{J\beta} - 2e^{-J\beta/2} \cos(\beta\varepsilon \sqrt{3}/2 - \pi/3) \end{aligned} \tag{4}$$

In the thermodynamic limit ($N \to \infty$) only the leading eigenvalue is relevant. However, due to the existence of level crossing, such an eigenvalue varies from λ_0 to λ_2 depending on the temperature and the values of the coupling constants J and ε. Crossing points of leading levels correspond to transition points of the models. The corresponding transition temperatures are given by

$$\beta_c = \frac{2(2m+1)\pi}{3\sqrt{3}\,\varepsilon} \tag{5}$$

m being a positive (negative) integer for positive (negative) values of the coupling ε. The lattice of transition points has a double-periodic structure; for fixed ε the period between transition temperatures is $\Delta_{\beta_c} = 4\pi/(3\sqrt{3}\,\varepsilon)$, and $\Delta_{\varepsilon_c} = 4\pi/(3\sqrt{3}\,\beta)$ is the period between values of ε at consecutive transition points for fixed β.

The periodic behavior in ε can easily be understood as a consequence of the invariance of the Hamiltonian under global spin rotations. In fact, the eigenvectors v_k, $k = 0, 1, 2$, of the transfer matrix $T(\beta, J, \varepsilon)$ for given values of the parameters β, J, and ε transform as

$$v'_k = e^{i2\pi m p_n/3} v_k = v_{(k+m, \mathrm{mod}\ 3)}$$

under spin rotations. Furthermore, since

$$T'(\beta, J, \varepsilon) = e^{i2\pi m p_n/3} T(\beta, J, \varepsilon)\, e^{-i2\pi m p_n/3} = T\left(\beta, J, \varepsilon + \frac{4\pi m}{3\beta\sqrt{3}}\right)$$

the corresponding eigenvalues $\lambda_k(\beta, J, \varepsilon)$, $k = 0, 1, 2$, are periodic on ε,

$$\lambda_{(k-m, \mathrm{mod}\ 3)}(\beta, J, \varepsilon) = \lambda_k\left(\beta, J, \varepsilon + \frac{4\pi m}{3\beta\sqrt{3}}\right)$$

The dependence of the energy density with respect to the temperature is discontinuous at the transition points (5),

$$\Delta\mathscr{E}\,|_{\beta=\beta_c} = \frac{3\varepsilon}{e^{3J\beta_c/2}+1} \tag{6}$$

which means that they correspond to first-order phase transitions. The energy gap at the transition points decreases as β_c increases and the transition becomes of second order at zero temperature ($\beta_c = \infty$) for any value of ε, as in the pure three-state clock model. The three-state model (2) is the simplest system of one-dimensional classical spin models undergoing a phase transition at nonvanishing temperature.

The existence of such a phase transition is due to the presence of the ε term in the Hamiltonian. For $\varepsilon = 0$, the Perron–Frobenius theorem (or any of the methods quoted in the first paragraph of this paper) implies that there is no phase transition, in this case, for any value of $\beta < \infty$. However, with $\varepsilon \neq 0$, the entries in the transfer matrix are not longer real (much less real and positive), so the Perron–Frobenius theorem cannot be applied, and, even in the absence of the exchange interaction ($J = 0$), the system undergoes a phase transition at the temperatures given by (5).

We remark that the Hamiltonian (2) involves only local (nearest-neighbor) interactions and although the ε term is imaginary, the transfer matrix is a self-adjoint operator. Furthermore, in chains with an odd number of sites the model satisfies the reflection-positivity property with respect to the center of the chain for any value of the coupling constants β and ε (see Appendix A). Reflection-positivity also holds in chains with an even number of sites for large values of β [$\beta > (2/3J)\log 2$] because then the transfer matrix is nonnegative (see Appendix A).

In the thermodynamic limit the behavior of the correlation function of spin variables $(\mathbf{s}_n)_k = (\cos(2k\pi p_n/q), \sin(2k\pi p_n/q))$

$$\langle (\mathbf{s}_n)_k \cdot (\mathbf{s}_{n+m})_{k'} \rangle = \frac{1}{2}\,\delta_{k'}^{-k}\left[\left(\frac{\lambda_{k_0-k}}{\lambda_{k_0}}\right)^m + \left(\frac{\lambda_{k_0+k}}{\lambda_{k_0}}\right)^m\right] \tag{7}$$

shows that at the transition points the correlation

$$\lim_{m\to\infty} \langle (\mathbf{s}_n)_k \cdot (\mathbf{s}_{n+m})_{k'} \rangle = \tfrac{1}{2}\delta_{k'}^{-k}\delta_{|k|}^{1} \tag{8}$$

becomes long-ranged. In fact, the correlation length diverges as $\xi(\beta) = (\beta^{-1} - \beta_c^{-1})^{-1}$ at the transition points. Therefore, the phase transitions of this model are very peculiar: they are of first order (for $\beta < \infty$), but have infinite correlation length. The critical exponents are $\nu = 1$ and $\eta = 1$. This interesting feature and other unusual properties of the model suggest that the conventional scenario for first-order transitions[13] does not hold in this case (see ref. 14 for criticism of first-order phase transition scenarios).

The 4-state model exhibits a similar behavior. In this case the transfer matrix has the following eigenvalues:

$$\begin{aligned} \lambda_0 &= 2\cosh J\beta + 2\cos\beta\varepsilon, & \lambda_1 &= 2\sinh J\beta + 2\sin\beta\varepsilon \\ \lambda_2 &= 2\cosh J\beta - 2\cos\beta\varepsilon, & \lambda_3 &= 2\sinh J\beta - 2\sin\beta\varepsilon \end{aligned} \tag{9}$$

Transition points correspond to crossings of leading levels. The transition temperatures can be read from the equations

$$\beta_c^+ = \frac{(4n+1)\pi}{4\varepsilon} + \frac{1}{\varepsilon}\sin^{-1}\left(\frac{e^{-J\beta_c^+}}{\sqrt{2}}\right) \tag{10a}$$

$$\beta_c^- = \frac{(4n+1)\pi}{4\varepsilon} + \frac{1}{\varepsilon}\cos^{-1}\left(\frac{e^{-J\beta_c^-}}{\sqrt{2}}\right) \tag{10b}$$

where n is an arbitrary integer and β^+ (β^-) corresponds to the crossing of levels with the same (opposite) index parity. In this case the lattice of transition temperatures is no longer periodic, unlike that of the 3-state model (5). However, for fixed temperatures β and exchange coupling J, the ε-lattice of transition couplings,

$$\begin{aligned} \varepsilon_c^+ &= \frac{(4n+1)\pi}{4\beta} + \frac{1}{\beta}\sin^{-1}\left(\frac{e^{-J\beta}}{\sqrt{2}}\right) \\ \varepsilon_c^- &= \frac{(4n+1)\pi}{4\beta} + \frac{1}{\beta}\cos^{-1}\left(\frac{e^{-J\beta}}{\sqrt{2}}\right) \end{aligned} \tag{11}$$

is periodic and its period, π/β, is larger than that of the 3-state model.

The transition is also of first order and the gap of the energy density at the transition points (10a) and (10b) is

$$\begin{aligned} \Delta\mathscr{E}^+ &= 2\varepsilon - \frac{\varepsilon - J + J\sin 2\beta_c\varepsilon}{\sin^2\beta_c\varepsilon} \\ \Delta\mathscr{E}^- &= 2\varepsilon - \frac{\varepsilon + J + J\sin 2\beta_c\varepsilon}{\sin^2\beta_c\varepsilon} \end{aligned} \tag{12}$$

respectively. The gap decreases as β_c increases, and vanishes in the limit $\beta_c \to \infty$, which corresponds to a second-order phase transition for any value of ε.

The reason for the existence of phase transitions is again the ε term. In fact, even in the pure ε case ($J=0$) there are phase transitions for the temperatures $\beta_c=(2n+1)\pi/2\varepsilon$. The 4-state model is reflection-positive for any temperature in odd chains (see Appendix). In even chains it is also reflection-positive for small temperatures $\beta>\sinh^{-1}(1/J)$, because then the transfer matrix is nonnegative.

The correlation function of spin variables $\mathbf{s}_n$ at the transition points (7) is constant in the thermodynamic limit (8). Therefore, the correlation length at such first-order transition points is infinite. The critical exponents are $\nu=1$ and $\eta=1$.

The general case of q-state models with $q>4$ has a similar behavior. The eigenvalues of the transfer matrix are

$$\lambda_k=\sum_{n=0}^{q-1}\exp[\beta J\cos(2\pi n/q)+i\beta\varepsilon\sin(2\pi n/q)-ik(2\pi n/q)] \tag{13}$$

Leading eigenvalues also cross an infinite number of times for finite temperatures. However, in this case there is not periodic behavior in the ε coupling. The transition points at finite temperature correspond to first-order phase transitions with an infinite correlation length ($\nu=1$, $\eta=1$). The only second-order transition occurs at zero temperature for vanishing ε. On the other hand, for a wide range of values of coupling parameters the transfer matrix is nonnegative and the models satisfy the reflection-positivity condition on any chain.

Let us now consider the continuum generalization ($q\to\infty$) of the above spin models. The fluctuating variable $\mathbf{s}$ sweeps in such a case the whole unit circle and the dynamics is defined by the Hamiltonian (2). The model is $O(2)$ invariant and it is the one-dimensional analog of the X–Y model. The partition function is defined by

$$Z(\beta,J,\varepsilon)=\prod_{m=1}^{N}\int_0^{2\pi}d\theta_m \times\prod_{n=1}^{N}\exp[\beta J\cos(\theta_n-\theta_{n+1})+i\beta\varepsilon\sin(\theta_n-\theta_{n+1})] \tag{14}$$

and we consider periodic boundary conditions. The corresponding transfer matrix is given by

$$[T\psi](\theta)=\int_0^{2\pi}d\theta'\,\{\exp[\beta J\cos(\theta-\theta')+i\beta\varepsilon\sin(\theta-\theta')]\}\,\psi(\theta') \tag{15}$$

and has the following eigenvalues:

$$\lambda_k = \begin{cases} 2\pi \left(\dfrac{J+\varepsilon}{J-\varepsilon}\right)^{k/2} I_{|k|}(\beta(J^2-\varepsilon^2)^{1/2}), & |\varepsilon| \leqslant J \\ 2\pi \left(\dfrac{J+\varepsilon}{\varepsilon-J}\right)^{k/2} J_{|k|}(\beta(\varepsilon^2-J^2)^{1/2}), & |\varepsilon| > J \end{cases} \tag{16}$$

where k is an integer and $I_{|k|}$, $J_{|k|}$ are the Bessel functions of integer order. The matching of the eigenvalues (16) at $\varepsilon = \pm J$ is completely smooth. All the eigenvalues λ_k are positive for $|\varepsilon| \leqslant J$ and reflection-positivity holds in such a case for any finite chain, whereas for $|\varepsilon| > J$ some of them might be negative and the system is only reflection-positive for odd chains.

We will restrict the analysis to the region $|\varepsilon| \leqslant J$, for simplicity, because the discussion for $|\varepsilon| \geqslant J$ is quite similar. The dependence of the eigenvalues (16) with respect to ε is displayed in Figs. 1 and 2 for $J=1$ and some values of β.

Transition points correspond to the crossing of two consecutive leading eigenvalues. Therefore the transition temperatures β_k are parametrized by an integer and given by the condition

$$\lambda_k(\beta_k, J, \varepsilon) = \lambda_{k+1}(\beta_k, J, \varepsilon) \tag{17}$$

where $\lambda_k(\beta, J, \varepsilon)$ is the leading eigenvalue of the transfer matrix for $\beta_{k-1} < \beta \leqslant \beta_k$. The values of ε corresponding to a transition point for fixed temperature and exchange coupling J are the solutions of the equation

$$\frac{I_{|k+1|}(\beta(J^2-\varepsilon_k^2)^{1/2})}{I_{|k|}(\beta(J^2-\varepsilon_k^2)^{1/2})} = \left(\frac{J-\varepsilon_k}{J+\varepsilon_k}\right)^{1/2} \tag{18}$$

The source of long-range fluctuations leading to the existence of phase transitions is again the ε term, which breaks the positivity-improving

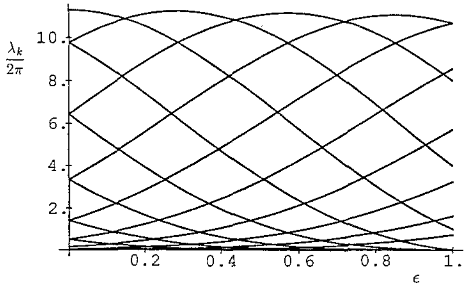

Fig. 1. Leading eigenvalues of the transfer matrix as functions of ε for $J=1$ and $\beta=4$.

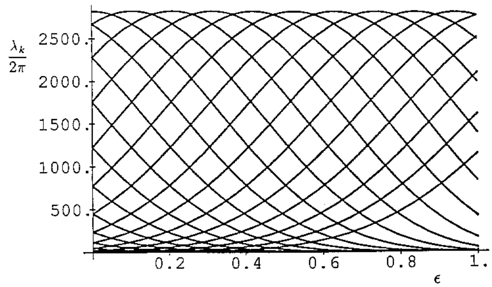

Fig. 2. The ε dependence of leading eigenvalues of the transfer matrix for $J=1$ and $\beta=10$. We remark the existence of a critical point at $\varepsilon=J=1$.

property of the transfer matrix. In the pure case $J=0$ the eigenvalues of the transfer matrix

$$\lambda_k = 2\pi J_{|k|}(\beta\varepsilon) \tag{19}$$

also exhibit crossing of leading levels indicating the existence of phase transitions.

We remark that for integer values of β, $\varepsilon_{\beta-1}=J$ is a transition point, as can be shown from the asymptotic expansion

$$\lambda_k(\beta, J, \varepsilon) = 2\pi \left(\frac{J+\varepsilon}{J-\varepsilon}\right)^{k/2} \left\{\frac{1}{\Gamma(k+1)}\left[\frac{\beta}{2}(J^2-\varepsilon^2)^{1/2}\right]^k + \mathcal{O}(J^2-\varepsilon^2)\right\} \tag{20}$$

for $\varepsilon \to J^-$. In the same way it can be shown that $\varepsilon_{1-\beta} = -J$ is a transition point for integer values of β.

The free energy density at the transition points (17) is discontinuous as for the case of the discrete spin models, indicating that the model undergoes a first-order phase transition. The corresponding energy density gap

$$\Delta\mathscr{E}_k = \frac{2}{J\beta}(\varepsilon\beta - k) - \frac{1}{J\beta} \tag{21}$$

decreases as $\beta \to \infty$ (see Fig. 3), but the limit gap vanishes only if $\varepsilon \to 0$.

The expectation value of the local observable $i\sin(\theta_n - \theta_{n+1})$, $\zeta(\beta, J, \varepsilon) = \langle i\sin(\theta_n - \theta_{n+1})\rangle$, is also discontinuous at the transition points. The corresponding gap is

$$\Delta\zeta(\beta, J, \varepsilon_k) = \frac{2\varepsilon_k(\varepsilon_k\beta - k) + \varepsilon_k - J}{\beta(J^2 - \varepsilon_k^2)} \tag{22}$$

Therefore, the different phases of the model can be characterized by the value of the order parameter $\zeta(\beta, J, \varepsilon_k)$. The physical interpretation of the existence of long-range fluctuations in this one-dimensional model is better understood in terms of such an order parameter. The dominant configura-

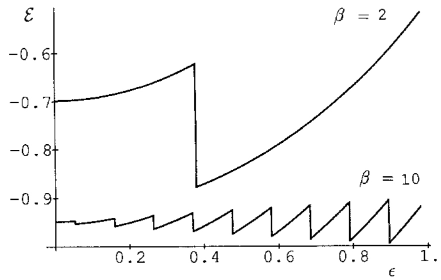

Fig. 3. Dependence of the free energy density on the ε parameter for $J=1$ and $\beta=2$, and 10.

tions of each phase are those whose spins turn around the periodic lattice chain forming a helix. The relative angle between nearest neighbor spins has a continuous dependence on ε except at the critical points, where it suffers a jump given by (22). It is remarkable that at the transition point both phases coexist and the model can have vortexlike excitation through the chain.

The correlation length of the finite-temperature transition points is infinite as in the case of clock models ($\nu=1$, $\eta=1$). There is only one second-order critical point at $\beta=\infty$, $\varepsilon=0$. The difference between the two types of critical points is that in the case of first-order transitions only the two highest level eigenvalues of the transfer matrix cross at the critical point, whereas in the case of second-order phase transition $\beta=\infty$, $\varepsilon=0$ all the levels degenerate to the leading eigenvalue. This implies that dilatation invariance only holds in the latter case, which would correspond to a fixed point of the renormalization group. A detailed analysis of the renormalization group flow of those systems will be carried out elsewhere. The continuum limit at the second-order critical point $\beta=\infty$, $\varepsilon=0$ can be taken keeping the leading eigenvalue of the transfer matrix fixed. It corresponds to the scaling limit in lattice spacing $a\to 0$ when $\beta\to\infty$ keeping βa and $\beta\varepsilon$ constants. The associated quantum system is a planar rotor. The scaling limit of the family of first-order transition points $\beta_k\varepsilon_k=k+1/2$ leads to a quantum system with degenerated ground state.[15]

In spite of the fact that this model has complex cloupings, the continuum limit leads to a consistent quantum mechanical system because the reflection-positivity property holds for large (small) values of β (ε). In fact, the corresponding Boltzmann weights are not effectively complex, because the partition function (14) can written as

$$Z(\beta, J, \varepsilon)=2\prod_{m=1}^{N}\int_0^{\pi} d\theta_m \prod_{n=1}^{N} e^{\beta J\cos(\theta_n-\theta_{n+1})}\cos[\beta\varepsilon\sin(\theta_n-\theta_{n+1})] \qquad (23)$$

where all Boltzmann weights are real. The only relevant features of the presence of ε terms are that the weights are not positive definite and that the system is not time-reversal invariant. The last property follows from the fact that reversing the orientation of the chain is equivalent to a change on the sign of the ε parameter and the spectrum of the transfer matrix is not invariant under such a change for $\varepsilon \neq 0$. This property is inherited by the corresponding quantum systems.

In summary, the families of one-dimensional models introduced above undergo first-order phase transitions due to the presence of a local imaginary ε term which breaks the positivity-improving character of the transfer matrix, but preserves in many cases reflection-positivity. The most relevant features of such phase transitions is that they have infinite correlation length and some other unusual properties,[16] which suggests that the conventional scenario for first-order phase transitions does not hold in such a case. The discussion of the peculiarities of this type of first-order transition will be carried out in a forthcoming paper.[16]

APPENDIX. REFLECTION-POSITIVITY

A1. Odd Chains

In chains with odd number of points the center of the chain belongs to the lattice. In such a case the chain can be split into two symmetric pieces $\Lambda_+ = \{(N+1)/2, (N+3)/2,..., N\}$ and $\Lambda_- = \{1, 2,..., (N+1)/2\}$ with nonempty overlapping. For any function F of spin variables with support on the sublattice Λ_+ its reflection θF with respect to the middle point $(N+1)/2$

$$\theta F(s_{(N+1)/2},..., s_1) = F(s_{(N+1)/2},..., s_N)$$

is supported on the lattice Λ_-. Reflection-positivity means that for any such function F the expectation value

$$\langle \overline{\theta F} F \rangle \geqslant 0 \tag{A.1}$$

is nonnegative. The overbar denotes complex conjugation and ordering reversal of spin variables,

$$\bar{G}(s_1,..., s_{(N+1)/2}) = G^*(s_{(N+1)/2},..., s_1) \tag{A.2}$$

The Hamiltonian of the q-state clock model (2) and its continuous generalizations splits into two pieces

$$\mathscr{H} = \mathscr{H}_+ + \mathscr{H}_- = \mathscr{H}_+ + \overline{\theta \mathscr{H}_+} \tag{A.3}$$

with support on Λ_+ and Δ_-, respectively. Thus, reflection-positivity (A.1) holds in those models for any value of β and ε, because

$$\langle \overline{\theta F} F \rangle = \prod_{n=1}^{N} \prod_{s_n} e^{-\beta \mathscr{H}_+} e^{-\beta \mathscr{H}_-} \overline{\theta F} F = \prod_{s_{(N+1)/2}} G^* G \geqslant 0 \tag{A.4}$$

with

$$G = \prod_{n=(N+1)/2}^{N} e^{-\beta \mathscr{H}_+} F$$

In fact, in this case reflection-positivity follows from the Hermitian character of the transfer matrix.

A2. Even Chains

If the chain has an even number of sites, its center does not belong to the chain and therefore it can be split into two disjoint pieces

$$\Lambda_+ = \{(N+2)/2, (N+4)/2,..., N\}, \qquad \Lambda_- = \{1, 2,..., N/2\}$$

The Hamiltonian of the clock models considered above can be split into three terms

$$\mathscr{H} = \mathscr{H}_+ + \mathscr{H}_- + \mathscr{H}_\pm = \mathscr{H}_+ + \overline{\theta \mathscr{H}_+} + \mathscr{H}_\pm \tag{A.5}$$

where $\mathscr{H}_\pm$ contains the terms coupling the spins $s_{N/2}$ and $s_{(N+2)/2}$. Proceeding in a similar way as in the previous case, we obtain

$$\begin{aligned} \langle \overline{\theta F} F \rangle &= \prod_{n=1}^{N} \prod_{s_n} e^{-\beta \mathscr{H}_+} e^{-\beta \mathscr{H}_-} \overline{\theta F} e^{-\beta \mathscr{H}_\pm} F \\ &= \prod_{s_{N/2}} \prod_{s_{(N+2)/2}} G^*(s_{N/2})\, T(s_{N/2}, s_{(N+2)/2})\, G(s_{(N+2)/2}) \\ &= (G, TG) \end{aligned} \tag{A.6}$$

for any function F of spin variables with support on Λ_+. The function G is defined by

$$G = \prod_{n=(N+2)/2}^{N} e^{-\beta \mathscr{H}_+} F$$

and $T(s_{N/2}, s_{(N+2)/2})$ is the transfer matrix element defined by

$$T(s_{N/2}, s_{(N+2)/2}) = \exp[-\beta \mathscr{H}_\pm (s_{N/2}, s_{(N+2)/2})]$$

Therefore in this case the reflection-positivity property of the models is equivalent to nonnegativity of the corresponding transfer matrix.

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